$$\int \frac{5x^2\,dx}{x^2+x+1} = \int 5 + \frac{-5x-5}{x^2+x+1}\,dx$$

$$\begin{array}{r|l} & \quad 5 \\ x^2+x+1 & \overline{\;5x^2} \\ & -(5x^2+5x+5) \\ \hline & \quad -5x-5 \end{array}$$

$$= \int 5\,dx - 5\int \frac{x+1}{x^2+x+1}\,dx$$

$$= 5x - \frac{5}{2}\int \frac{2(x+1)}{x^2+x+1}\,dx$$

$$= 5x - \frac{5}{2}\int \frac{2x+2}{x^2+x+1}\,dx$$

$$= 5x - \frac{5}{2}\int \frac{2x+1+1}{x^2+x+1}\,dx$$

$$= 5x - \frac{5}{2}\int \frac{2x+1}{x^2+x+1} + \frac{1}{x^2+x+1}\,dx$$

$$= 5x - \frac{5}{2}\ln|x^2+x+1| - \frac{5}{2}\int \frac{1}{x^2+x+1}\,dx$$

Now to tackle $\int \frac{1}{x^2+x+1}\,dx$

$$\int \frac{1}{x^2+x+1}\,dx$$

$$= \int \frac{1}{(x+1/2)^2+3/4}\,dx$$

hmm... $(x+1/2)^2$

$= x^2+x+\frac{1}{4}$

let $u = x+1/2$

$du = dx$

$$= \frac{1}{2}\int \frac{1}{u^2+3/4}\,2du$$

let $w = 2u$ $\quad u = \frac{w}{2}$

$dw = 2du$

$$= \frac{1}{2}\int \frac{1}{(\frac{w}{2})^2+3/4}\,dw$$

$$= \frac{1}{2}\int \frac{1}{\frac{w^2}{4}+3/4}\,dw$$

$$= \frac{1}{2}\cdot\frac{1}{1/4}\int \frac{1}{w^2+3}\,dw$$

$$= 2\cdot\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{w}{\sqrt{3}}\right)+C$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2u}{\sqrt{3}}\right)$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2(x+1/2)}{\sqrt{3}}\right)$$

#21 $\displaystyle\int \frac{x^2-2x-2}{x^3-1}\,dx$

$$\frac{x^2-2x-2}{x^3-1} = \frac{x^2-2x-2}{(x-1)(x^2+x+1)}$$

$$\frac{x^2-2x-2}{(x-1)(x^2+x+1)} = \frac{A}{(x-1)} + \frac{Bx+C}{x^2+x+1}$$

$$A(x^2+x+1) + (Bx+C)(x-1) = x^2-2x-2$$

$$Ax^2 + Ax + A + Bx^2 - Bx + Cx - C = x^2 - 2x - 2$$

solve $\begin{cases} A+B=1 \\ A-B+C=-2 \\ A-C=-2 \end{cases}$

$$\begin{array}{r} A+B=1 \\ 2A-B=-4 \\ \hline 3A=-3 \end{array}$$

$\boxed{A=-1}$ $\boxed{B=2}$ $\boxed{C=1}$

$$\frac{x^2-2x-2}{(x-1)(x^2+x+1)} = \frac{A}{(x-1)} + \frac{Bx+C}{x^2+x+1}$$

So our integral is

$$\int \frac{-1}{x-1} + \frac{2x+1}{x^2+x+1}\,dx$$

$$\int \frac{-1}{x-1} + \frac{2x+1}{x^2+x+1}\,dx$$

$$= -\ln|x-1| + \ln|x^2+x+1| + C$$

$$(1-x)(1+x) = 1-x^2$$

$$(1-x)(1+x+x^2) = 1-x^3$$

$$(1-x)(1+x+x^2+x^3) = 1-x^4$$

$$(1-x)(1+x+x^2+\dots+x^n)$$

$$= 1-x^{n+1}$$

$$\begin{array}{r}
1+x+x^2+x^3 \\
1-x \\
\hline
-\cancel{x}-\cancel{x^2}-\cancel{x^3}-x^4 \\
1+\cancel{x}+\cancel{x^2}+\cancel{x^3} \\
\hline
1-x^4
\end{array}$$

$$\frac{1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x-1)}$$

$$\frac{1}{(x^2+1)x^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Dx+E}{x^2+1}$$

$$\frac{4}{(2x+3)^2(x^2+5)} = \frac{A}{2x+3} + \frac{B}{(2x+3)^2} + \frac{Cx+D}{x^2+5}$$

$$\frac{1}{(2x-1)(x^2+x+1)} = \frac{A}{2x-1} + \frac{Bx+C}{x^2+x+1}$$